

How to estimate the interaction parameter of a weak quantum measurement

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There exists a shared basis for parameter estimation and the emergence of weak values: both work on the initialisation of the system, a weak interaction with a probe, and a final readout stage. However, there is no established framework encompassing this relation at the most general level. Here we establish such connections in a universal theory, and illustrate our results for the specific case of single-photon polarisation.

Quantum metrology is concerned with the optimal estimation of an extremely weak quantum interaction from the statistics obtained by appropriate combination of input states and output measurements^{1,2}. Such faint quantum interactions can also be used to perform statistical measurements with negligible back-action, where the result of the weak measurement conditioned by a final output measurement gives rise to weak values for the physical properties of quantum systems³. These measurement values have attracted particular attention because they appear to exceed the limits set by the eigenvalues obtained in precise measurements with stronger interactions.

Although quantum metrology and weak measurements are both based on a similar sequence of quantum input, weak effect, and output measurement⁴, the fundamental relation between the sensitivity of parameter estimates and the observation of weak values is still somewhat unclear. Here, we clarify the connection by explaining how the results of weak measurements can be used to estimate the interaction parameter describing the coupling between the system and the probe. Significantly, we find that weak values provide an optimal estimate of the system property that determines the interaction, resulting in maximal sensitivities for a much wider range of output measurements than one would expect from classical considerations.

Let us consider the elements involved in weak quantum measurements³ as shown in Fig. 1 and realized in a number of recent experiments^{5–11}. A quantum system described by a quantum state $|\psi\rangle$ interacts weakly with a well known probe state. Since the effect of this coupling on the probe depends on the value of a system property \hat{A} , information about \hat{A} can be extracted from a measurement on the probe. Specifically, the probability w_m of an outcome m will depend on the value of \hat{A} . As for any quantum measurement, the interaction between system and probe also induces an uncontrollable disturbance in the state of the system. However, this disturbance is negligibly small if the interaction is very weak. It is therefore possible to define the state of the system more pre-

cisely by postselecting the result of an additional output measurement $|f\rangle$. This double definition of the quantum state by preparation and postselection results in the weak values $\langle\hat{A}\rangle_{\text{wv}}$, which can be far outside the range of the eigenvalues observed in strong measurements.

On closer inspection, the effect of the weak measurement interaction is determined by the product of a small interaction parameter ϵ with the system property \hat{A} . In the following, we consider a situation where the interaction parameter is unknown and we wish to estimate the small value of ϵ from the measurement outcomes m by using known combinations of input and output states for the system. Significantly, the best estimation of the system property \hat{A} is obtained by using the weak values $\langle\hat{A}\rangle_{\text{wv}}$ predicted from the known input-output combination. The accuracy of this estimate will directly affect the sensitivity of the estimate of the unknown interaction parameter ϵ . Quantum metrology can thus provide a practical test of the accuracy of weak values.

To simplify the discussion of the physics involved in the weak measurement, we use a formulation that treats

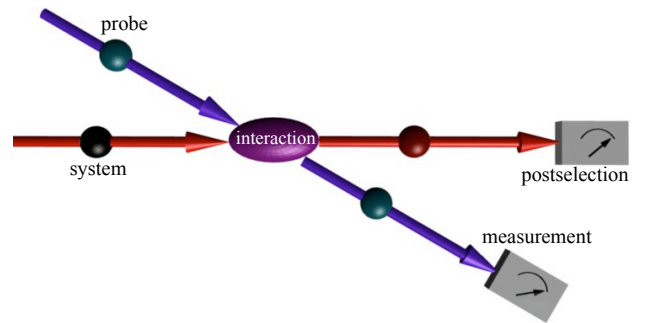


FIG. 1: A weak measurement on a system is obtained by weakly coupling its observable \hat{A} to a probe state, which is then measured. Weak values emerge when considering the outcomes of the measurement conditioned by the postselection of a particular outcome determined in a final measurement on the system.

the preparation and measurement of the probe as integral parts of a single measurement operator associated with the measurement outcome m . For a weak interaction with the property $\hat{\mathcal{A}}$ of the system, this self-adjoint operator can be written as

$$\hat{E}_m = \sqrt{w_m} (\hat{I} + \epsilon \kappa_m \hat{\mathcal{A}}). \quad (1)$$

Here, κ_m represents the correlation between the outcome and the value of $\hat{\mathcal{A}}$ and w_m is the probability of m without the interaction. We notice the formal similarity to a weak unitary transformation, which is obtained when ϵ is replaced by an imaginary phase parameter. As pointed out in Ref.⁴, phase estimation in quantum interferometry is then equivalent to an estimation of the interaction parameter from imaginary weak values.

The experimental results of a weak measurement are characterized by the joint probabilities $p(m, f)$ of obtaining a weak measurement result m and observing an output state $\{|f\rangle\}$ of the system. According to quantum mechanics, this probability is given by $|\langle f | \hat{E}_m | \psi \rangle|^2$, which includes the measurement back-action as a quadratic term in ϵ and in $\hat{\mathcal{A}}$. For sufficiently weak interactions, this term can be neglected and the joint probabilities of the weak measurement are given by

$$p(m, f) = w_m |\langle f | \psi \rangle|^2 \left(1 + 2\epsilon \kappa_m \text{Re} \left(\frac{\langle f | \hat{\mathcal{A}} | \psi \rangle}{\langle f | \psi \rangle} \right) \right). \quad (2)$$

In this expression, the effects of $\hat{\mathcal{A}}$ on the quantum statistics is given by a single value equally conditioned by initial and final state. This value is the real part of the weak value of $\hat{\mathcal{A}}$ obtained by postselection of a final state $|f\rangle$ (Ref.³).

If we consider the estimation of the interaction parameter ϵ from measurements with known input states $|\psi\rangle$, the weak value tells us how the effect of the interaction on the output statistics depends on the final measurement outcome f . In the formalism of quantum metrology¹², this dependence is expressed by the logarithmic derivatives,

$$\left. \frac{\partial}{\partial \epsilon} \ln(p(m, f)) \right|_{\epsilon=0} = 2\kappa_m \text{Re} \left(\frac{\langle f | \hat{\mathcal{A}} | \psi \rangle}{\langle f | \psi \rangle} \right). \quad (3)$$

Thus, the weak values of $\hat{\mathcal{A}}$ provide a fundamental mathematical expression for the sensitivity of the output statistics to small changes in the interaction parameter ϵ . Specifically, the logarithmic derivatives determine the Fisher information, which is the inverse lower bound of the estimation error σ_ϵ^2 . By normalizing the values of κ_m to $\sum_m w_m \kappa_m^2 = 1$, the Fisher information (and hence the sensitivity of the parameter estimation) can be expressed as¹²

$$\begin{aligned} F &= \sum_{m,f} p(m, f) (\partial_\epsilon \ln(p(m, f))|_{\epsilon=0})^2 \\ &= 4 \sum_f p(f) \text{Re} \left(\frac{\langle f | \hat{\mathcal{A}} | \psi \rangle}{\langle f | \psi \rangle} \right)^2. \end{aligned} \quad (4)$$

Since the sensitivity is given by the average of the squared real parts of the weak values obtained for different postselected outcomes f , optimal results are obtained for final measurements with completely real weak values. In this case, $p(f) = |\langle f | \psi \rangle|^2$ can be eliminated and the summation results in $F = 4 \langle \psi | \hat{\mathcal{A}}^2 | \psi \rangle$. Therefore, the maximal sensitivity is determined by the average value of $\hat{\mathcal{A}}^2$ in the input state, and different measurement strategies for the postselection of f merely result in different distributions of the weak values. Specifically, anomalous weak values much larger than the maximal eigenvalues of the observables do contribute more to the sensitivity, but this effect is compensated by the relatively low probability $p(f)$ of such outcomes.

We can now illustrate the principles explained above using the data obtained in our experimental weak measurement on the polarization of single photons. For this experiment, we produced photon pairs by spontaneous parametric down conversion in a bismuth borate nonlinear crystal. The pump beam is a frequency-doubled, pulsed Ti:Sa laser laser ($\lambda_p = 820\text{nm}$, $\Delta t = 100\text{fs}$, repetition rate 82 MHz, average power $P_p = 50\text{mW}$). One of these photons is used as the test system s , while the other acts as a measuring probe p . The interaction between them is obtained by a photonic controlled-sign (c-s) gate based on a single partially polarizing beam splitter with transmittivity $\eta_V = 1/\sqrt{3}$ ($\eta_H = 1$) for the vertical, V , (horizontal, H) polarization^{13–15}.

The operation of the gate changes the sign of the $|V, V\rangle_{p,s}$ component of the quantum state, where both the signal and the probe photon are vertically polarized. Thus, the interaction distinguishes between the horizontal and vertical polarization states of the signal and can be used to realize a weak measurement of the Stokes parameter $\hat{S}_{HV} = |H\rangle\langle H| - |V\rangle\langle V|$. Since the interaction also requires a vertical polarization component in the probe, weak interactions can be realized by using probe states close to the horizontally polarized state. For a probe in a superposition state given by $|H\rangle_p + \epsilon |V\rangle_p$, the initial polarization is slightly biased towards the diagonal polarization D . However, an interaction with a vertically polarized system photon changes the bias to the opposite diagonal polarization A . Therefore, a measurement of the diagonal output polarization realizes the measurement operation^{5,6} given by Eq.(1), with $\hat{\mathcal{A}} = \hat{S}_{HV}$, $\kappa_D = -\kappa_A = 1$, and $w_D = w_A = 1/2$.

Post-selection of the s photon is then performed by a destructive measurement in the diagonal basis ($f=D$ or $f=A$). Here we focus on the statistics originated by such an operation by observing the conditional probabilities on a particular post-selection event, and by using the measurement data for an estimate of the interaction parameter. Since the measurement interaction is responsible for the difference between the two post selected outcomes $p(D|f)$ and $p(A|f)$, it is possible to obtain a statistical estimate of ϵ for each postselected set of output

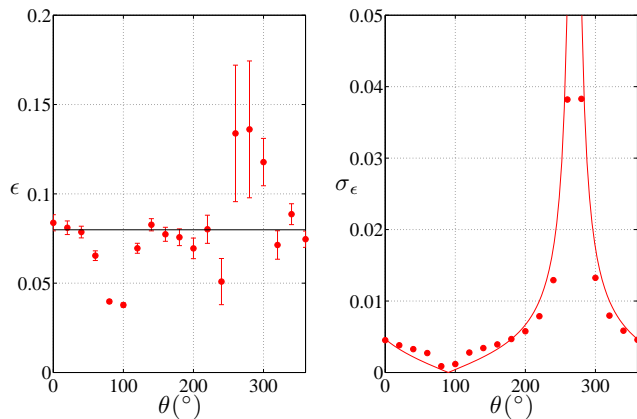


FIG. 2: Experimental results for the estimate of the interaction parameter using the postselected probabilities with $f=A$. Left panel: estimate of the value of ϵ as a function of the input state $|s\rangle = \cos \frac{\theta}{2}|H\rangle + \sin \frac{\theta}{2}|V\rangle$; the solid line indicates the set value, red dots the experimental data. Right panel: statistical uncertainties σ_ϵ ; the points represent the experimental uncertainty due to Poissonian noise on the count rates, the solid line is the estimation based on Eq. (2).

measurements using

$$\epsilon = \frac{p(D|f) - p(A|f)}{2\langle \hat{S}_{HV} \rangle_{wv}}. \quad (5)$$

For this estimate, the statistical error is given by the binomial distribution of outcomes between $m = A$ and $m = D$. We have evaluated this error and confirmed that its inverse corresponds to the contribution to the total Fisher information from the outcome f in Eq. (4). The results of this analysis are shown in Fig. 2. In the right panel we plot the estimate of epsilon based on the postselected events with $f=A$: the discrepancies are to be attributed both to experimental imperfections of the gate, and to the breakdown of the linear approximation Eq. (5). Despite these imperfections, the uncertainties follow the trend described by Eq. (2). It is evident how the sensitivity of the postselected estimation attains a maximum around $\theta=90^\circ$. How can we understand such behaviour?

An insight is provided by the inspection of anomalies in the weak values. According to Eq. (3), the weak values for a final polarization measurement f can be determined from the change in output probabilities caused by differential changes in the interaction parameter ϵ ,

$$\langle \hat{S}_{HV} \rangle_{wv} = \frac{1}{2} \frac{\partial}{\partial \epsilon} \ln(p(D|f)) = -\frac{1}{2} \frac{\partial}{\partial \epsilon} \ln(p(A|f)). \quad (6)$$

We confirmed this relation for a postselection of anti-diagonal polarization $f = A$ on input states with variable linear polarization by approximating the derivative as a finite differential between a low coupling of $\epsilon=0.08$ and zero coupling and taking the average of the two values obtained for the two meter outcomes m . The results,

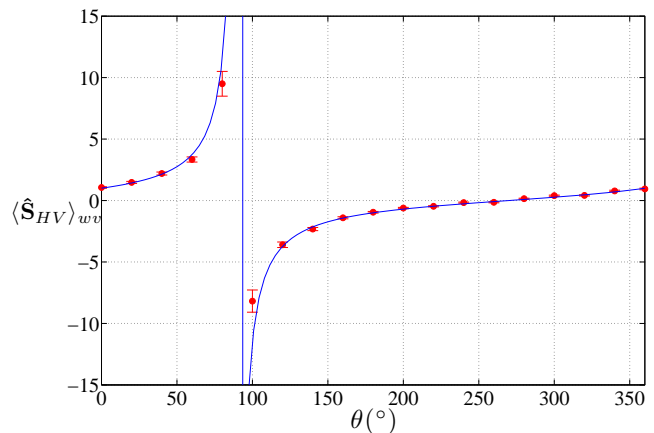


FIG. 3: Evaluation of the weak values from the logarithmic derivatives of the output probabilities as a function of input state polarization. θ is the orientation of polarization on the Poincaré sphere, where $\theta = 0$ corresponds to horizontal polarization and $\theta = 180^\circ$ corresponds to vertical polarization. The data was obtained for a postselection of the anti-diagonal output polarization A , corresponding to an angle of $\Theta = 270^\circ$. Red dots show the results obtained from the experimental data with error bars representing Poissonian statistical errors. The solid line shows the predicted weak values.

which are in good agreement with the predicted weak values, are shown in Fig. 3.

We now need to compare these results with an analysis of the Fisher information, as given by Eq. (4). In the region around $\theta=90^\circ$, where anomalous weak values can be observed, most of the Fisher information originates from the rather low number of postselected events with weak values far greater than the maximal eigenvalues of ± 1 . In contrast, the majority of “normal” outcomes carry almost no information about the interaction parameter ϵ . We plot the Fisher information as a function of θ in Fig. 4. We show the total Fisher information and also the contribution to the Fisher information from the postselected state $|A\rangle_s$. Note that postselecting on $|D\rangle_s$ would produce a complementary curve to the one shown. Furthermore, the behaviour of the uncertainties in Fig. 2 originates from the fact that weak values determine the distribution of sensitivity between the postselected outcomes.

To understand the significance of the result, it is important to consider the relation between statistics and weak values in more detail. While it seems natural that extreme weak values result in higher sensitivities, it is not at all clear, why the interaction strength should depend on the choice of the final measurement f . Indeed, the present results indicate that the sensitivity originates from the initial fluctuations of \hat{S}_{HV} , as evidenced by the constant total sensitivity obtained from both outcomes f . In this sense, weak values simply represent an estimate of the unknown value of \hat{S}_{HV} in $|\psi\rangle$. Interestingly, this interpretation also results in a paradox: if the goal

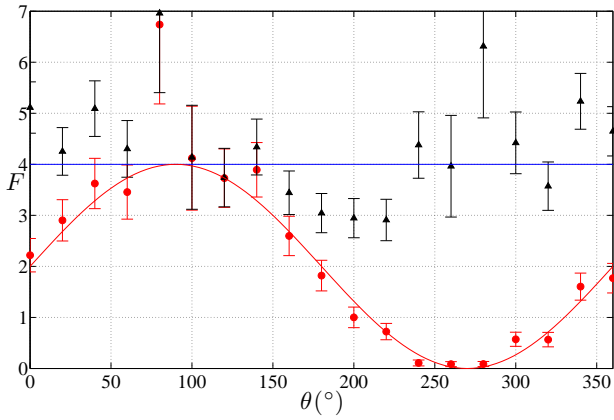


FIG. 4: Experimental results for the Fisher information. Red dots show the results for the contribution associated only with $|A\rangle_s$; the red solid line shows the theoretical predictions for the ideal case. Error bars are estimated from the Poissonian statistics of the coincidence counts, resulting in larger uncertainties for anomalous values. The black triangles show the experimental values for the total Fisher information including both post-selection events.

is an optimal estimate of \hat{S}_{HV} in $|\psi\rangle$, best results should be obtained from postselections of eigenstates of \hat{S}_{HV} . Since weak values are not consistent with the outcomes

of strong measurements, one would expect an additional error when weak values are used to estimate the value of \hat{S}_{HV} . However, the present results indicate otherwise. For the purpose of estimating the interaction parameter, weak values clearly represent an optimal estimate of the system property in question. Consequently, there is a rather surprising freedom of choice in the selection of the final measurement used to determine the output state of the system. The estimation of interaction parameters using weak measurement thus reveals an amazing flexibility in the way that quantum mechanics distributes the available information between physical properties, with both practical and fundamental implications for the way we think about the counterintuitive properties of quantum systems.

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